Effective modal masses

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1 Abstract

Usual dynamic analysis techniques express structure deformation in a new base of rigid modes and elastic modes. Each mode is characterised by a modal mass and effective masses associated to different directions. The normalisation of these modes is arbitrary, this imply that they have no physical meaning when considered alone. On the other hand, the effective mass has a physical meaning and allow, in a lot of cases, simplification of the computation of deformations and stresses. They allow a trade-off of useful modes and simple computations on complex structures.

Indeed, each mode can be interpreted as a mass-spring-dashpot system oriented in the rigid modes space along a specific direction. The mass in this case is the effective mass. An elastic mode will be excited by a junction degree of freedom (dof) acceleration {a> if the projection of {a> along the mode specific direction is non-null. When a force is applied on the internal dof's, each elastic mode will be excited by the projection of the force on this mode. Moreover, the reaction at the junction dof implied by a mode will have a direction parallel to the mode specific direction.

This paper outlines the real representation that can be attributed to the effective masses and to these directions. A new normalisation of these modes is proposed in order to give them a physical significance. We introduce also modal stresses to allow easy computation of dynamic stresses.

2 Introduction

The use of modal analysis for structure analysis leads to very interesting and useful results. Nevertheless, for large structures or to obtain important accuracy, the necessary computation time is very important.

A better understanding of the representation by modes can improve the analysis by a better selection of the useful modes and a better estimation of the results expected [1], [2], [3], [5].

In this paper, a new representation of matrices and vectors is proposed. Explanations are proposed in appendix (§9.1)

3 Reminder of modal analysis

The equation we want to solve is equation (1).

(1) $$\begin{vmatrix} [M] & [M\} \\ \{M] & \{M\} \end{vmatrix} \begin{vmatrix} [\ddot{q}\rangle \\ \{\ddot{q}\rangle \end{vmatrix} + \begin{vmatrix} [C] & [C\} \\ \{C] & \{C\} \end{vmatrix} \begin{vmatrix} [\dot{q}\rangle \\ \{\dot{q}\rangle \end{vmatrix} + \begin{vmatrix} [K] & [K\} \\ \{K] & \{K\} \end{vmatrix} \begin{vmatrix} [q\rangle \\ \{q\rangle \end{vmatrix} = \begin{vmatrix} [F\rangle \\ \{F\rangle \end{vmatrix}$$

Commonly, we define a base of modes. Some of them are considered as rigid modes as they imply no structure deformation (in isostatic or rigidly mounted cases). As a normalisation, the displacement of the degrees of freedom (dof) corresponding to junction nodes are set to one in these modes. We consider here applications where the support is infinitely rigid or isostatic. In those cases, the rigid modes (m = 6) will be defined as $[\phi\} = - [K]^{-1}[K\}$ if $[K]^{-1}$ exists. In this case, we can also observe that $\{K\}-\{K][K]^{-1}[K\}$ is null. The rigid modes can be written as (2),

(2) $$|\phi\} = \begin{vmatrix} [\phi\} \\ \{I\} \end{vmatrix}$$

with {I} = {diag (...1…)}.

The elastic modes [ϕ>> can be extracted from equation (3):

(3) $$[\ [-\mathrm{diag}(\omega_k^2)][M] + [K]\ ]\ [\phi\rangle\rangle = 0 \qquad k = 1,2,\ldots n$$

The normalisation of these modes is arbitrary. For each mode we associate a frequency (3), a modal mass (4), a modal stiffness (5), a modal structural damping ($\varepsilon_k = c_k/2m_k\omega_k$) and a vector linking this mode to the rigid modes (6).

(4) $$m_k = \langle\phi_k]\ [M]\ [\phi_k\rangle$$

(5) $$k_k = \langle\phi_k]\ [K]\ [\phi_k\rangle$$

(6) $$\{L_k\rangle = \{\ \phi\ |\ M\ |\ \phi_k\ \rangle$$

The resolution of the equation using these parameters gives the following results.

(7) $$[q_0\rangle = [\ [\phi\}+[\phi\rangle\rangle\langle\langle\mathrm{diag}(\ldots\omega^2 H_k/k_k\ldots)\rangle\rangle\langle\langle L\}\ \ \}\{q_0\rangle + [\phi\rangle\rangle\langle\langle\mathrm{diag}(\ldots H_k/k_k\ldots)\rangle\rangle\langle\langle\phi]\ [F_0\rangle$$

and

(8) $$\{R_0\rangle = \{-\{L\rangle\rangle\langle\langle\mathrm{diag}(\ldots\omega^2 H_k/k_k\ldots)\rangle\rangle\langle\langle\phi] - \{\phi]][F_0\rangle + \{\ -\omega^2\{L\rangle\rangle\langle\langle\mathrm{diag}(\ldots\omega^2 H_k/k_k\ldots)\rangle\rangle\langle\langle L\} - \omega^2\{\phi M\phi\} + \{\phi K\phi\}\ \ \}\ \{q_0\rangle$$

with the dynamic amplification factor (here written H_k),

(9) $$H_k(\omega) = \frac{1}{1 - \frac{\omega^2}{\omega_k^2} + 2\,i\,\varepsilon_k\,\frac{\omega}{\omega_k}}$$

In (7), the first term $[\phi\}\ \{q_0\rangle$ represents the rigid displacement; the second term represents the displacement due to junction dof displacement and

the third term represents the displacement due to the forces applied to the internal dof. In (8), the first term represents the reaction forces due to the force applied to the internal dof, the second term represents the reaction forces due to the junction dof displacement.
Within the parameters, the frequency and the damping are independent of the mode normalisation and thus are the only parameters that have a physical significance.
By combining them, we can define one effective mass matrix for each mode (10).

(10) $$\{M_k\} = \{L_k> 1/m_k <L_k\} \qquad k = 1,2,\ldots n$$

These matrices are now independent of modal normalisation and have a physical significance. By definition, only 6 elements of this matrix are independent, the elements outside main diagonal being a combination of the diagonal ones. There is also only one non null eigenvalue with one corresponding eigenmode. This eigenvalue is the trace of the matrix, the mode is parallel to vector $\{L_k>$.
The main difficulty of this approach is the fact that the mass matrix combines different kinds of elements: mass ones (expressed in kg) and inertia ones (expressed in kg.m²). As a result, the modal mass, the effective mass, … are also combinations of mass and inertia.
In order to avoid this problem, we will assume that the modes have units: displacements in meters and rotations in radians. By this way, the modal masses as well as the effective mass matrices components are all expressed in kg.m².

4 Effective parameters

Let's define the effective mass $m_{eff,k}$ as the eigenvalue of the effective mass matrix. We propose to modify the modes normalisation in order that the norm of $\{L_k>$ equals $m_{eff,k}$.
The solution is to multiply all modes by the square root of the effective mass divided by the modal mass.

(11) $$[\phi_{eff,k}> = [\phi_{e,k}> . \sqrt{\frac{m_{eff,k}}{m_k}}$$

This will result in a new set of modes with modal masses equal to effective masses, $\{L_k>$ vector as a norm of $m_{eff,k}$ and a direction parallel to the direction of the eigenmode of the k^{th} effective mass matrix and a new modal stiffness appears that will be called effective stiffness (12).

(12) $$k_{eff,k} = m_{eff,k} . \omega_k^2$$

Those new modes are called effective modes.

5 Resolution of equations with this new base

5.1 Imposed acceleration along junction dof

When the $\{a_0>$ acceleration is imposed at the junction dof, from equation (8), we can extract the unknown junction reaction $\{R_0>$:

(13) $$\{R_0> = \sum_k T_k(\omega)\, m_{eff,k} .\alpha_{al_k} a_0 \{l_{eff,k}> + \{MB\}\{a_0>$$

where a_0 is the acceleration vector norm, $\{l_{eff,k}>$ is the unitary vector in direction $\{L_{eff,k}>$, α_{alk} is the cosine of the angle between acceleration vector and each $\{l_{eff,k}>$ vector and T_k is given by:

(14) $$T_k(\omega) = 1 + \left(\frac{\omega}{\omega_k}\right)^2 H_k(\omega)$$

In (13), we can see that for each mode, the acceleration is projected on corresponding $\{l_{eff,k}>$. This projected acceleration (acc) excites the mass-spring-dashpot system that gives a force $T_k(\omega).m_{eff,k}$.acc (the response of a 1-dof system in the $\{l_{eff,k}>$ direction.
The acceleration can also be projected along the eigenmodes of {MB}. Along each of these direction, a mass, whose value is the corresponding eigenvalue, is accelerated and also transmit a force to the base.
The sum of all these forces is the resulting reaction force.
An identical computation can show that the displacement of the structure.

(15) $$[q_0> = \sum_k \frac{H_k(\omega)}{\omega_k^2} .\alpha_{al_k} a_0 [\phi_{eff,k}> + [\phi\ \}\left\{\frac{a_0}{\omega^2}\right.>$$

Once again, equation (15) shows that the acceleration is projected along $\{l_{eff,k}>$ and the displacement of the mass defines now the coefficient to apply to the effective modes.

5.2 Force imposed on internal dof

If we apply forces on the internal dof, we can extract the junction reaction from (8).

(16)
$$\{R_0> = \sum_k \frac{\omega^2}{\omega_k^2} H_k(\omega) . < \phi_{eff,k}\ ][F_0>.\{l_{eff,k}> - \{\phi\ ][F_0>$$

So to obtain the solution, we can again separate in forces ($<\phi_{eff,k}][F_0>$) applied at each mass. The reaction force is applied along $\{l_{eff,k}>$. The sum of all forces is the reaction at the junction dof.
The displacement are expressed by (17).

(17) $$[q_0> = \sum_k \frac{H_k(\omega)}{k_k} . < \phi_{eff,k}\ ][F_0>.[\phi_{eff,k}>$$

As in the previous case, the displacement of the masses is the coefficient to apply to the effective modes.

6 Physical meaning

6.1 Effective modes

The k^{th} effective mode is associated to a specific direction defined by $\{L_k>$ in the rigid modes space of the structure. The effective mode $[\phi_{eff,k}>$ is the resulting deformation, multiplied by a factor ω_k^2, when a uniform unitary acceleration is imposed along direction $\{L_k>$ at the junction dof.
If we compute the deformation energy in this case we have:

$$(18) \quad \frac{1}{2}\frac{<\phi_{eff,k}]}{\omega_k^2}[K]\frac{[\phi_{eff,k}>}{\omega_k^2}=\frac{1}{2}\frac{k_{eff,k}}{\omega_k^4}=\frac{1}{2}\frac{m_{eff,k}}{\omega_k^2}$$

6.2 Effective masses

The effective mass $m_{eff,k}$ of mode k is the fraction of the total static mass that can be attributed to this mode (static inertia for rotation modes). As shown by (18), this also represents the energy absorbed by this mode multiplied by $2.\omega_k^2$ when nominally excited by a static unitary acceleration at the junction nodes. The $m_{eff,k}$ units are in any case kg.m².

6.3 Effective participation factors

The $\{L_{eff,k}>$ vectors define a set of directions in the rigid mode space (in general a 6-dimension space).
Each effective mode can be represented by a mass-spring-dashpot system oriented along $\{l_{eff,k}>$ in the rigid modes space. The mass value is $m_{eff,k}$, the spring constant is $k_{eff,k}$ and the damping coefficient is unchanged. The elastic behaviour of the structure can be represented in the rigid modes space by this set of 1-D mass-spring-dashpot systems (Fig 6.1).

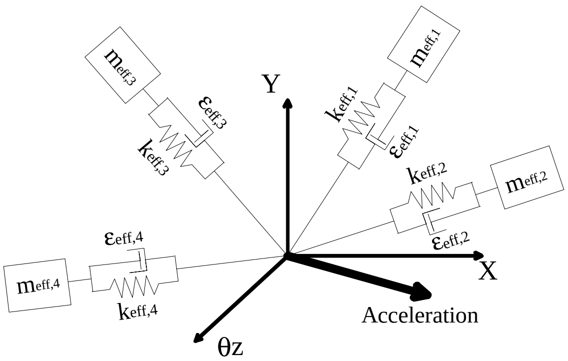

Figure 6-1: Representation of elastic modes in rigid modes space (X, Y, q_z)

To obtain the mass associated to a direction, we can project the effective mass along this vector. In our case, it is simply the $\{L_k>$ components (L_i along rigid mode direction i). To obtain the mass associated with a given direction i, when the system is excited along any direction j, we come back to the effective mass matrix definition: the component $(\{L_k>1/m_{eff,k}<L_k\})_{i,j}$ is the mass along rigid mode direction i when the junction dof j is excited.

6.4 {MB} matrix, junction mass

The {MB} matrix represents the masses (kg.m²) attributed to the junction nodes. This matrix is null in continuous system. In equation (13), {MB} can be expanded in its eigenvalues and eigenmodes. So, we can represent it in the rigid mode space by masses defined by the eigenvalues of the matrix and positioned along the eigenvectors of the matrix. These masses are connected to the centre of the frame by rigid wires.

6.5 Overall structure representation

The complete structure can be represented in the rigid modes space by all mass-spring-dashpot systems and {MB} masses.

6.6 Guyan matrix

The Guyan matrix is the sum of all effective mass matrices and the {MB} matrix. It can also be represented by m masses, equal to the eigenvalues of the matrix and positioned along the eigenvectors of the matrix.
It can be easily demonstrated that the trace of Guyan matrix is equal to the sum of the effective masses and the trace of the MB matrix.

$$(19) \qquad \sum_k m_{eff,k} + tr\{MB\} = tr\{M0\}$$

In our rigid modes space, it means that the sum of all masses (effective and {MB} masses) is equal to the sum of {M0} masses.

6.7 Junction dof acceleration

An acceleration of the junction dof $\{a_0>$ can also be represented in this space by a vector. As shown by (13), this acceleration will excite each k_{eff} spring system and each {MB} mass by means of its projection along $\{l_{eff,k}>$ and eigenvectors of {MB}. The resulting force given by all these 1-D systems is equivalent in direction (in the rigid modes space) and in intensity to the real reaction force in the junction dof $\{q_0>$ of the structure.
Equation (15) shows that the relative displacement of each mass (with respect to the origin of the axes) is the coefficient to apply to the modes to define the deformation of the structure (note that no displacement is possible along {MB} eigenvectors).
Moreover the displacement of the masses is equivalent to the displacements of the centres of gravity associated to the effective modes.

6.8 Internal dof force

A force applied to the internal dof's will be represented in the effective modes base $[\phi_{eff,k}»$. In the set of 1-D spring systems, each mass will be submitted in its own direction by a force $<\phi_{eff,k}][F_0>$. The resulting force (or torque) of all those effective masses corresponds in the rigid modes space to the real force in the structure junction dof's. The displacement is once again the coefficient to apply to each effective mode $[\phi_{eff,k}>$.

7 Stresses computation [4]

We define ‡σ(t)> as a vector of selected linear combination of the stress tensor components calculated or interpolated at any point of the structure. The size of this vector is arbitrary ($N \neq n$). The equation

(20) $$‡\sigma(t)> = ‡S|\ |q(t)>$$

simply states that this linear combination of the stress tensor elements is a linear relation of displacements in the approximation of the linear finite element model, assuming small deformations.
‡S| is a Nx(m + n) matrix. Using equation (7), we can write

(21) $$‡\sigma>= ‡S]\ [\phi»«\mathrm{diag}(\ldots\omega^2 H_k/k_k\ldots)»«L\}\ \{q_0> + ‡S]\ [\phi»«\mathrm{diag}(\ldots H_k/k_k\ldots)»«\phi]\ [F_0>$$

The first term is the dynamic stresses produced by the junction movement $\{q_0>$ and the second term represents the dynamic stresses resulting from the internal forces $[F_0>$.
Similarly to the effective elastic modes, we define the *effective modal stresses* with:

(22) $$‡\sigma_{eff,k}> = ‡\sigma_{e,k}> \sqrt{\frac{m_{eff,k}}{m_k}}$$

By this way, the stresses in the structure can be obtained by equation (15) where a_0 is the acceleration and again the displacement of the 1-D effective masses are the coefficients to apply to the effective modal stresses.

8 Conclusion

We propose a general representation of a structure by a set of 1-D mass-spring-dashpot systems in the space of rigid modes.
The effective masses distribution can easily characterise the relative contribution of the modes when the structure is excited along a direction with a given frequency.
For example this representation can be useful to evaluate the structure response to vibration tests where rigid bases are commonly used.

9 Appendix

9.1 Generalised Dirac notation

Structural analyses lead to use non square matrices of various sizes: m x n, (m + n) x n, ... In order to clearly identify the matrix size, we will use an extension of the Dirac notation. In this formalism, the symbols (Table 9-1) are written before and after the matrix name to identify the number of lines (left symbol) and columns (right symbols). It means that the matrix |A| is a (m+n)x(m+n) symmetric matrix, [p> is a n components column vector, <q} is a m components line vector, |B? is a (m+n)x(m+n) non symmetric matrix.

Left	Right	Size	Left	Right	Size
|	|	m + n	<	>	1 (one)
[	]	n	«	»	n (non symmetric)
{	}	m	?	?	m + n (non symm.)
			‡	‡	N (non symmetric)

Table 9-1: matrix symbols

When two matrices are associated, adjacent symbols must be identical. The meaning of this association is the product of the 2 matrices. When several matrices are multiplied, the final result has the size determined by the external symbols (for example <A| |B} {C] [D> is the product of 4 matrices and the result is a scalar).
The transposition operation consists in inverting the symbols $[A\}^T = \{A]$. When the matrix is composed of sub-matrices, this operation is defined by inverting all symbols and transposing the main matrix.

(23) $$\begin{vmatrix} [A] & [B\} \\ \{C] & \{D\} \end{vmatrix}^T = \begin{vmatrix} [A] & [C\} \\ \{B] & \{D\} \end{vmatrix}$$

10 References

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